Condensed group cohomology and duality

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Motivation

In arithmetic geometry, we are often interested in studying schemes over a ring R. A scheme generalizes the concept of an algebraic variety, which is useful to give a geometrical structure to the solutions of systems of polynomial equations over a ring R. A useful way to study the geometry of a scheme X over R is to compute its *cohomology groups*, namely algebraic objects (abelian groups) $\mathrm{H}^n(X, A)$ for integers n and "coefficients" A.

The conjectural picture (Geisser-Morin, [1])

Considering separated schemes X over the ring of integers $\mathcal{O}_K$ of a p-adic field K, there exists a *topological* cohomology theory, i.e. groups $\mathrm{H}^q(X, A)$ which are not merely algebraic objects, but also locally compact topological spaces. Here A is a locally compact abelian group of finite rank with a possible "twist". If we consider the generic fiber X_K, these cohomology groups should satisfy a duality of locally compact abelian groups. This means that we should obtain $\mathrm{H}^q(X_K, A')$, for "related" A' and A, as the Pontryagin dual of $\mathrm{H}^{2d-q}(X_K, A)$, where d is the dimension of the scheme X. It is important that not only the algebraic structures of these groups are related, but also their topologies.

In my case, that is the conjectural picture with $d = 1$ and $X = \mathrm{Spec}(\mathcal{O}_K)$, the cohomology of X_K coincides with the *condensed group cohomology* of the Weil group W_K. This is an algebraic object associated to the field K, naturally provided with a topology.

1. Group Cohomology

If G is a group acting on an abelian group A, we can define

$$C^i(G, A) = \mathrm{Maps}(G^i, A), \;\; d^i : C^i(G, A) \to C^{i+1}(G, A)$$

where d^i is a morphism of abelian groups which depends on the action of G on A. We define the abelian group $\mathrm{H}^q(G, A)$ as the q-th cohomology group of the complex

$$\dots \to C^{i-1}(G, A) \to C^i(G, A) \to C^{i+1}(G, A) \to \dots$$

It can be defined, equivalently, as q-th right derived functor of

$$(-)^G : G-\mathbf{Mod} \to \mathbf{Ab}, \qquad A \mapsto A^G$$

This categorical definition allows the cohomology theory to behave well with respect to the change of coefficients. For example, if we have a short exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

we get a long exact sequence of abelian groups

$$\dots \to \mathrm{H}^q(G, A) \to \mathrm{H}^q(G, B) \to \mathrm{H}^q(G, C) \to \mathrm{H}^{q+1}(G, A) \to \dots$$

The topological case

If G is a topological group acting continuously on a topological abelian group A, it is possible to define *continuous group cohomology* $\mathrm{H}^q_{cont}(G, A)$ by replacing $\mathrm{Maps}(G^i, A)$ with $\mathrm{Cont}(G^i, A)$ in the above definition. However, this does not come from a categorical definition, hence it doesn't behave well with respect to the change of coefficients. This problem can be solved by means of *condensed mathematics*.

2. Condensed Mathematics

What is condensed mathematics?

Condensed Mathematics is a theory, developed by Dustin Clausen and Peter Scholze, which aims to replace topological spaces by *condensed sets*, a category having much more favorable properties. This would allow us to do algebra with a topology, which was not possible before (e.g., topological abelian groups are not an abelian category).

From the categorical point of view $\mathbf{Cond}(\mathbf{Set})$ is very similar to $\mathbf{Set}$, but it also sees topological phenomena. This is expressed by the following

Theorem([2, Proposition 1.7, Theorem 2.2])

- There is a fully faithful embedding $\underline{(-)} : \mathbf{Top}^{\mathbf{cg}} \hookrightarrow \mathbf{Cond}(\mathbf{Set})$, where $\mathbf{Top}^{\mathbf{cg}}$ is the category of "nice" topological spaces (compactly generated). This embedding respects limits, and hence algebraic structures.
- $\mathbf{Cond}(\mathbf{Ab})$ is an abelian category satisfying the same Grothendieck axioms as $\mathbf{Ab}$.

Hence "nice" topological groups embed in condensed groups, "nice" topological abelian groups embed in condensed abelian groups... We can do algebra with a topology!

Condensed Group Cohomology

Given a topological group G acting continuously on a topological abelian group A, we get a condensed group $\underline{G}$ acting on a condensed abelian group $\underline{A}$ and we have a functor

$$\underline{\Gamma}(\underline{G}, -) : \mathbf{Cond}(\underline{\mathbf{G}} - \mathbf{Mod}) \to \mathbf{Cond}(\mathbf{Ab}), \;\; M \mapsto \mathrm{Hom}_{\mathbb{Z}[\underline{G}]}(\mathbb{Z}, M)$$

- If G, A are discrete, $\underline{\Gamma}(\underline{G}, \underline{A}) = \underline{A^G}$ with the discrete topology.
- More generally, if G, A have a "nice" topology, $\underline{\Gamma}(\underline{G}, \underline{A}) = \underline{A^G}$ with the subspace topology of A.

We define *condensed cohomology groups* $\underline{\mathrm{H}}^q(G, A)$ as $R^q\underline{\Gamma}(\underline{G}, \underline{A})$. They are better behaved than "continuous cohomology groups" defined above since they are defined as derived functors. Not only they are abelian groups, but they are *condensed*, hence they can have a topology: this is a property of the conjectured cohomology groups.

3. The cohomology of the Weil group

Let $K/\mathbb{Q}_p$ a finite extension, and k its residue field. The Weil group W_K is a topological group which is a modification of its absolute Galois group G_K.

- The Weil group $W_k \cong \mathbb{Z}$ is the subgroup of $G_k \cong \hat{\mathbb{Z}}$ generated by the Frobenius.
- The Weil group W_K is the topological pullback of W_k under the surjection $G_K \to G_k$
$$W_K := G_K \times_{G_k} W_k.$$

For any continuous W_K-topological abelian group A, we have condensed abelian groups

$$\underline{\mathrm{H}}^q(W_K, A) := R^q\underline{\Gamma}(\underline{W_K}, \underline{A}),$$

and we can define "dual objects" $\underline{A}^D$ and the corresponding cohomology groups

$$\underline{\mathrm{H}}^q(W_K, A^D).$$

The duality

The duality of the conjectural picture is related to this question: for which A is the cup-product

$$\underline{\mathrm{H}}^q(W_K, A) \times \underline{\mathrm{H}}^{2-q}(W_K, A^D) \longrightarrow \mathbb{R}/\mathbb{Z}$$

a perfect pairing?

4. Our goals

My research is focused on the properties of condensed group cohomology, and in particular of the Weil group W_K of a finite extension of $\mathbb{Q}_p$.

Understanding condensed group cohomology

- To establish the relation between condensed cohomology and continuous cohomology.
- To determine the topology of the cohomology groups when possible.

The case of the Weil group

We are trying to find topological W_K-modules A such that

1. the cohomology groups $\underline{\mathrm{H}}^q(W_K, A)$ and $\underline{\mathrm{H}}^q(W_K, A^D)$ are locally compact abelian groups,
2. the pairing
$$\underline{\mathrm{H}}^q(W_K, A) \times \underline{\mathrm{H}}^{2-q}(W_K, A^D) \longrightarrow \mathbb{R}/\mathbb{Z}$$
is perfect and realises $\underline{\mathrm{H}}^q(W_K, A^D)$ as the Pontryagin dual of $\underline{\mathrm{H}}^{2-q}(W_K, A)$.

Then, we want to relate this duality with the duality of the conjectural picture.

Some results:

- For profinite G and $A = \mathbb{R}$, or A "solid", the complex computing the condensed group cohomology is the "condensed version" of the complex computing continuous cohomology.
- For compact G, the cohomology groups are discrete if A is discrete.
- (In progress) If the underlying topological abelian group is a "strongly finite type" locally compact abelian group with finite p torsion, 1. and 2. are true under a hypothesis on the action of W_K (the inertia group of K acts via a finite quotient).
- For $A = \mathbb{R}/\mathbb{Z}$ and $q = 1$, the duality would give the isomorphism
$$K^\times \xrightarrow{\sim} (\underline{\mathrm{Hom}}(W_K, \mathbb{R}/\mathbb{Z}))^\vee = W_K^{ab}$$
of local class field theory "à la Weil".

References

[1] Thomas H. Geisser and Baptiste Morin. Pontryagin duality for varieties over p-adic fields, 2021.

[2] Peter Scholze. Lectures on condensed mathematics, 2019.